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Letter to the Editor

Vibration of an annular membrane attached to a free, rigid core

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1. Introduction

The vibration of circular and annular membranes is fairly well known [1]. However, there seems to be little literature on the coupling between membranes and rigid bodies. The present Letter considers such a problem: an annular membrane with a free, massed, finite core. We assume the membrane tension is large enough such that both the weight of the core and the weight of the membrane cause only small static deformations, and the vibration problem becomes decoupled. We ask, what is the effect of the core on the natural frequencies of the membrane? Notice the core can be any axisymmetric mass, including a cylinder, a sphere or a disk.

When the radius of the core shrinks to zero, the problem becomes a circular membrane supporting a point mass at the center. If further the mass is relatively large, it becomes a pinpoint constraint [2], and if the mass is relatively small, it resembles a hard spot [3]. Since for both special cases the frequency is the same as that of a circular membrane without a core, would a finite point mass also have no effect on the frequency?

2. Formulation

Fig. 1(a) shows an annular membrane of radius R attached to a rigid cylinder (or any axisymmetric body) of radius bR. Normalize all lengths by R. Let the (transverse) displacement be given by

$$u = w(r)\sin(n\theta)e^{i\Omega t}, \tag{1}$$

where w is an amplitude function, r, θ are cylindrical co-ordinates, t is the time and Ω is the frequency of oscillation. The equation of motion of the membrane is the Helmholtz

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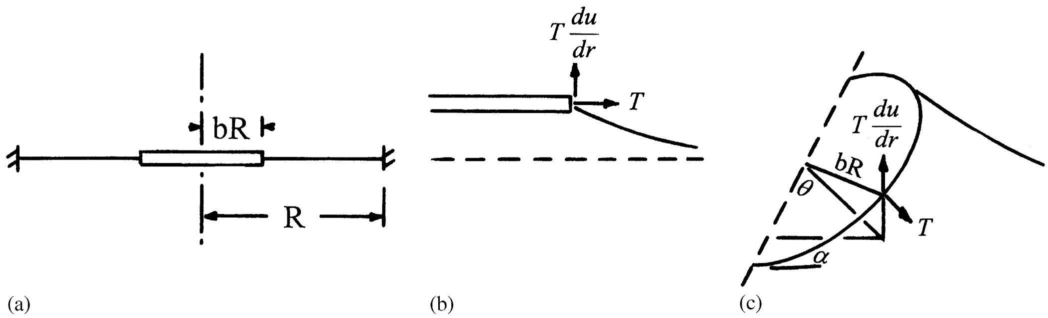

Fig. 1. (a) The circular membrane with a rigid core. (b) Forces in symmetric vibration. (c) Forces in antisymmetric vibration.

equation [1]

$$w''(r) + \frac{w'(r)}{r} - n^2\frac{w(r)}{r^2} + k^2 w = 0, \tag{2}$$

where $k = \Omega R$ [(density per area ρ)/(tension per length T)]$^{1/2}$ is the normalized frequency. The boundary conditions are that the displacement is zero at the outer boundary

$$w(1) = 0 \tag{3}$$

and that the dynamic loads are balanced with those of the rigid core.

The rigid core would sustain only two kinds of transverse vibrations: an up–down symmetric vibration and an antisymmetric rotational vibration about a diameter through the center of mass. The symmetric case ($n = 0$) is shown in Fig. 1b, where a vertical force balance gives

$$2\pi\, bT\frac{\partial u}{\partial r}(b) = m\frac{\partial^2 u}{\partial t^2}(b). \tag{4}$$

Here m is the mass of the solid core. Eq. (1) then gives

$$2bw'(b) + k^2\sigma w(b) = 0, \tag{5}$$

where

$$\sigma = m/\pi R^2\rho \tag{6}$$

is a ratio representing the relative mass of the solid. Fig. 1c shows the antisymmetric case ($n = 1$). A moment balance about the swivel axis $\theta = 0$ gives

$$2\int_0^\pi \left[Tb^2R\sin\theta\frac{\partial u}{\partial r}(b) - TbR\sin\theta u(b)\right]\mathrm{d}\theta = I\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}, \tag{7}$$

where I is the moment of inertia of the solid about the swivel axis and α is the inclination angle related to u by

$$\alpha = \frac{u(b)}{bR\sin\theta}. \tag{8}$$

Substitution of Eq. (1) and simplifying gives

$$b^3w'(b) - (k^2\eta - b^2)w(b) = 0, \tag{9}$$

where

$$\eta = \frac{I}{\pi R^4 \rho} \tag{10}$$

is a ratio representing the relative moment of inertia of the solid.

The solid core does not accommodate vibrations with higher nodal diameters ($n \geqslant 2$). The corresponding boundary condition for the membrane is thus

$$w(b) = 0, \quad n \geqslant 2. \tag{11}$$

3. Some asymptotic properties

For the axisymmetric case the solution to Eq. (2) satisfying Eq. (3) is

$$w = \mathrm{Y}_0(k)\mathrm{J}_0(kr) - \mathrm{J}_0(k)\mathrm{Y}_0(kr), \tag{12}$$

where J and Y are Bessel functions. Eq. (5) then gives the characteristic equation

$$2b[\mathrm{Y}_0(k)\mathrm{J}_1(kb) - \mathrm{J}_0(k)\mathrm{Y}_1(kb)] - k\sigma[\mathrm{Y}_0(k)\mathrm{J}_0(kb) - \mathrm{J}_0(k)\mathrm{Y}_0(kb)] = 0. \tag{13}$$

For small arguments the Bessel functions have the following expansions:

$$\mathrm{J}_0(z) = 1 + \mathrm{O}(z^2), \quad \mathrm{Y}_0(z) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{z}{2}\right)\right] + \mathrm{O}(z^2),$$

$$\mathrm{J}_1(z) = \frac{z}{2} + \mathrm{O}(z^3), \quad \mathrm{Y}_1(z) = -\frac{2}{\pi z} + \frac{1}{\pi} z \ln z + \mathrm{O}(z). \tag{14}$$

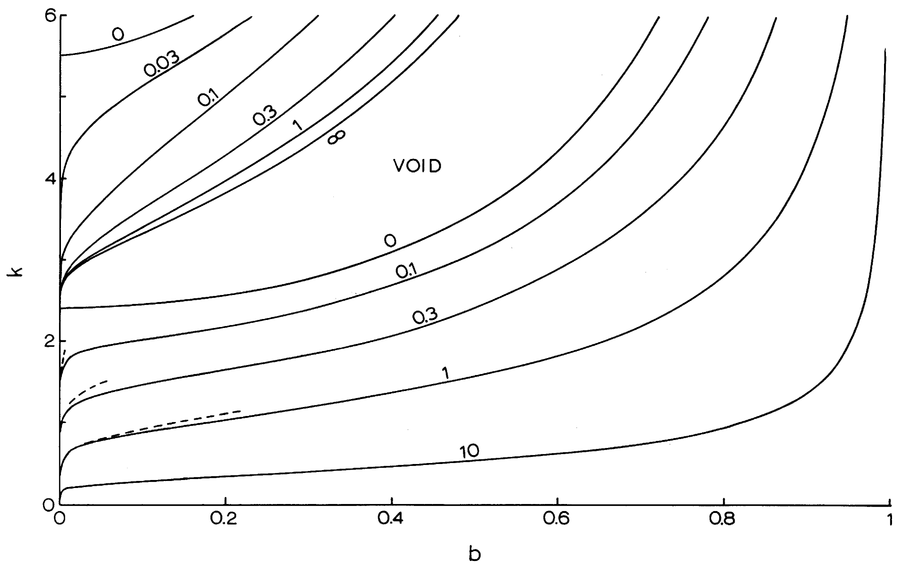

Fig. 2. Frequencies for the symmetric case for constant σ values. Dashed curves are from Eq. (16).

Thus for small b, The leading terms of Eq. (13) simplifies to

$$J_0(k)(2 + k^2\sigma \ln b) = 0. \tag{15}$$

There are two sets of roots to Eq. (15). The first set consists of the zeroes of J_0 which govern the symmetric vibration of a circular membrane *without* a core. This result is independent of the core mass, although higher order corrections would be affected. The second set has lower frequency, and is given by

$$k = \sqrt{\frac{2}{\sigma|\ln b|}} \tag{16}$$

which becomes zero as b approaches zero. Notice that if σ is infinite (a pinned center), this frequency is identically zero and should be discarded since it yields a trivial solution for displacement. On the other hand, if σ is zero (the collared case), the frequency is infinite and also insignificant. Only for finite mass do we get the second set of frequencies.

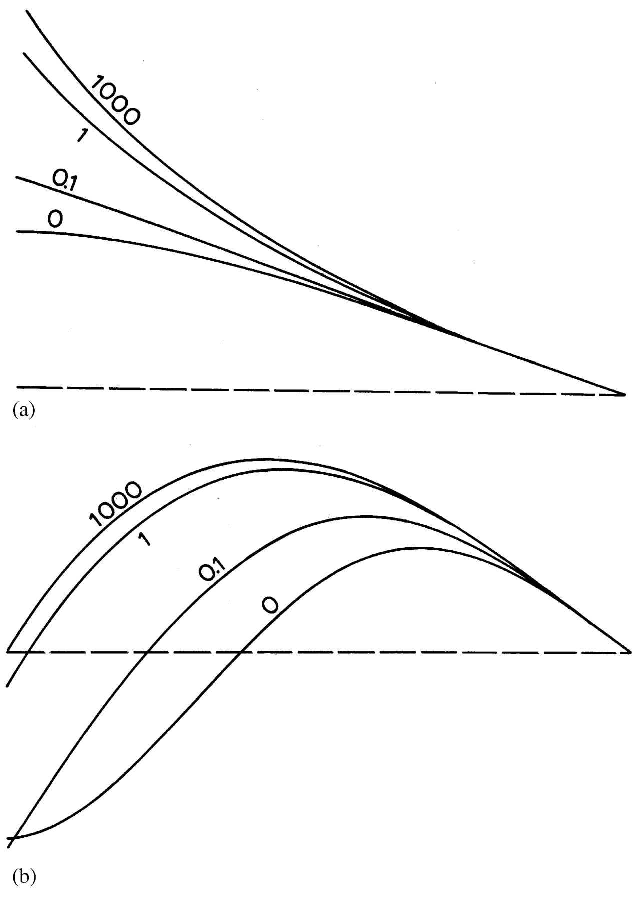

Fig. 3. Eigenfunctions for symmetric case, $b = 0.2$. σ values are shown. (a) First mode, (b) second mode.

For the antisymmetric ($n = 1$) case the general solution is

$$w = Y_1(k)J_1(kr) - J_1(k)Y_1(kr). \tag{17}$$

Eq. (9) yields

$$kb^3[Y_1(k)J_0(kb) - J_1(k)Y_0(kb)] + (k^2\eta - 2b^2)[Y_1(k)J_1(kb) - J_1(k)Y_1(kb)] = 0. \tag{18}$$

Similar asymptotic expansions for small b show

$$J_1(k)(k^2\eta - 2b^2) = 0. \tag{19}$$

The first factor is recognized as the antisymmetric mode of the circular membrane, while the second factor gives a new frequency important only for finite inertia

$$k = b\sqrt{\frac{2}{\eta}}. \tag{20}$$

The higher modes for $n \geqslant 2$ yield the characteristic equation

$$Y_n(k)J_n(kb) - J_n(k)Y_n(kb) = 0, \tag{21}$$

which is the same as that of an annular membrane with fixed boundaries. Since the core mass has no effect, we shall not investigate further.

4. Results and discussion

For given core radius b, mass ratio σ or moment of inertia ratio η, the exact non-linear characteristic equations, Eqs. (13) and (18), are solved for the frequency k by a simple root search

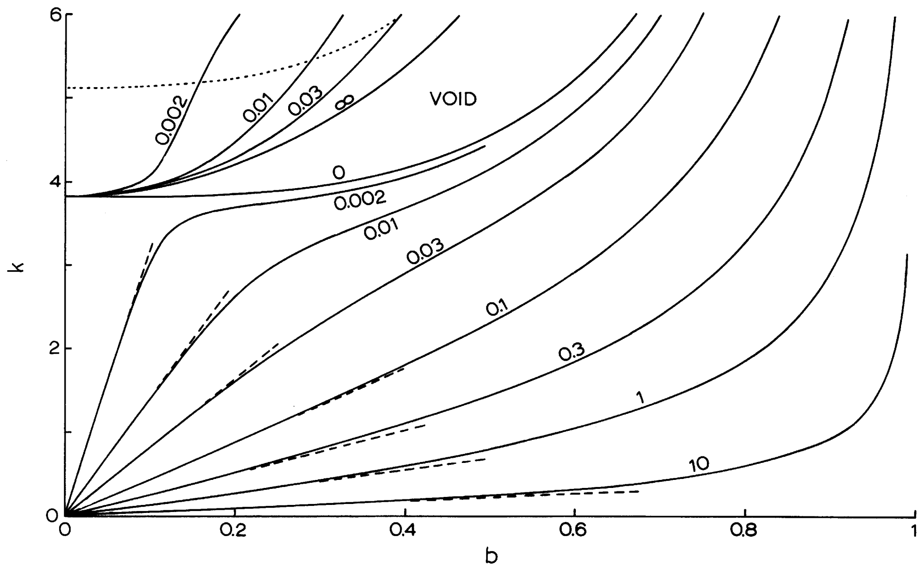

Fig. 4. Frequencies for the antisymmetric case for constant η values. Dashed lines are from Eq. (20). Dotted line is from Eq. (21) for $n = 2$.

algorithm. Fig. 2 shows k, for the axisymmetric case, as a function of b and various constants σ. The curve $\sigma = \infty$ corresponds to the annular membrane with fixed boundaries, whose singular behavior for small b (sharply rising from 2.4048, the first zero of J_0) was discussed previously [2]. The curves for $\sigma = 0$ represent the collared membrane (annular membrane bounded by a mass-less inner ring). The fundamental frequency also starts from 2.4048, but rises very slowly [3]. For finite mass ratios, we were expecting the frequencies to lie between these two limiting curves. But we are surprised to see that there is a void (given b, some frequencies are absent for any mass ratio) between these two curves. Thus the frequencies for finite mass ratios are essentially separated into distinct bundles. As b is increased, the lowest frequency bundle, called the first mode, rises steeply from zero as $|\ln b|^{-1/2}$ according to Eq. (16), then becomes infinite as b approaches one. The effect of increasing mass ratio σ is to decrease the frequency. This lowest bundle, however, does not include the singular cases of the mass-less collar ($\sigma = 0$) or the fixed boundary ($\sigma = \infty$). The second bundle, or second mode, starts from 2.4048 and increases with b as $|\ln b|^{-1}$. It does not include the $\sigma = 0$ case but includes the $\sigma = \infty$ case studied by Wang [2]. Typical eigenfunctions for the first two modes are shown in Fig. 3. Keep in mind that the first mode does not exist for $\sigma = \infty$, the amplitudes are arbitrary.

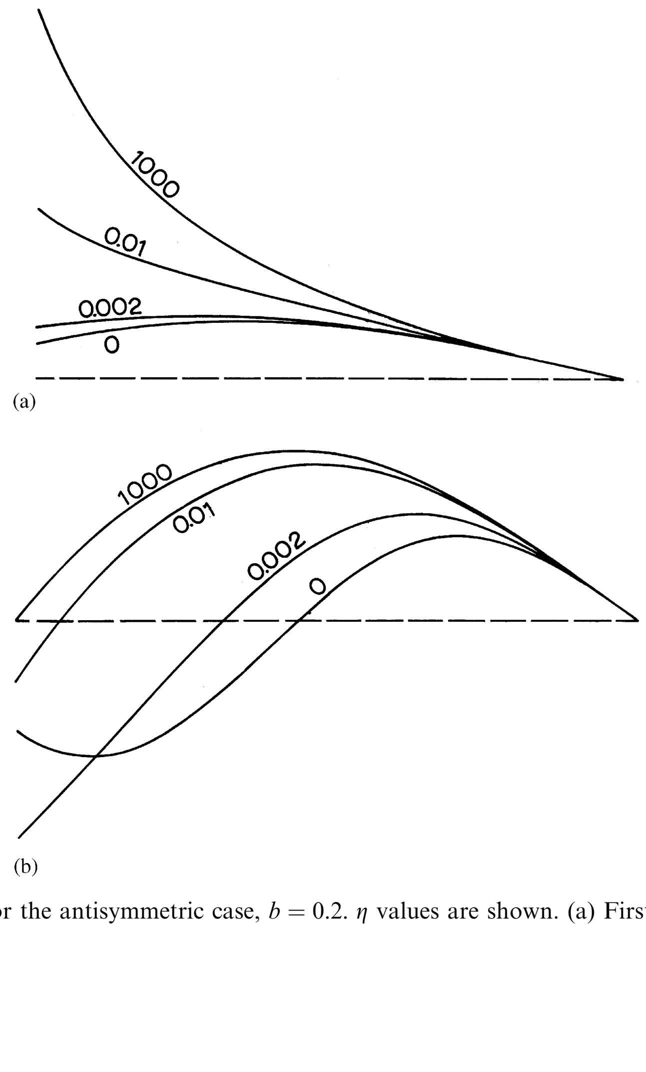

Fig. 5. Eigenfunctions for the antisymmetric case, $b = 0.2$. η values are shown. (a) First mode, (b) second mode.

Table 1
Ratio of η/σ; A is the aspect ratio of the finite cylinder

Geometry	Disk	Ring	Solid sphere	Solid cylinder
η/σ	$b^2/4$	$b^2/2$	$2b^2/5$	$b^2(4A^2+3)/12$

Table 1
Ratio of η/σ; A is the aspect ratio of the finite cylinder

The results for the antisymmetric case are shown in Fig. 4. Now the first mode increases from zero proportional to b as indicated by Eq. (20). Again the singular cases for the moment of inertia ratios $\eta = 0$ or $\eta = \infty$ are excluded. The second mode, separated by a void region, starts from 3.8317, the first zero of $J_1(k)$, and can be shown to increase as the square of b. Typical eigenfunctions for the antisymmetric case are shown in Fig. 5. Notice the sensitivity to low η values.

For a given rigid core and membrane density, one can use the values of $\sigma$and η to find the frequencies of vibration from Figs. 2 and 4. The ratio (η/σ) is independent of membrane properties and is given in Table 1 for some simple geometries.

Since $b<1$, the value of η is in general much less than that of σ, or the axisymmetric case mostly dominates the fundamental frequency. However, for a core which has relatively large moment of inertia, such as the solid cylinder of high aspect ratio A, the antisymmetric (wobbling) case dominates. Also, due to its slower rise from zero, the (first) antisymmetric frequency is always lower than that of the symmetric frequency at small b values.

If the dimension of the mass is indeed identically zero, our results show that the only frequencies are the zeroes of $J_n(k)$, or the mass has no effect on the membrane. Since all mass has some dimension, we see from Fig. 2 that for the symmetric case there is a sharp rise, including a new low frequency first mode. This singular behavior at small b is difficult to predict by any energy method or numerical method. The finiteness of the dimension also contributes to some moment of inertia. Fig. 4 shows that although the antisymmetric case has no singularity at $b = 0$, the small moment of inertia still has considerable effect, especially promoting the low frequency wobbling mode.

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